

Gauge Dependence in the Nonlinearly Realized Massive $SU(2)$ Gauge Theory

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Abstract

The implementation of the 't Hooft α -gauge in the symmetrically subtracted massive gauge theory based on the nonlinearly realized $SU(2)$ gauge group is discussed. The gauge independence of the self-mass of the gauge bosons is proven by cohomological techniques.

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1 Introduction

A consistent subtraction scheme for massive non-Abelian gauge theories based on a nonlinearly realized gauge group has been recently proposed in [1]. The symmetric subtraction algorithm was already successfully applied to the four-dimensional nonlinear sigma model in the flat connection formalism in Refs.[2]-[5].

The Feynman rules of the nonlinearly realized massive gauge theory entail that already at one loop level there is an infinite number of divergent amplitudes involving the pseudo-Goldstone fields ϕ_a [1]. The latter amplitudes are uniquely fixed by implementing a defining local functional equation [1, 2] which encodes the invariance of the path-integral Haar measure under local $SU(2)_L$ transformations¹

$$\begin{aligned}\Omega' &= U_L \Omega, \\ A'_\mu &= U_L A_\mu U_L^\dagger + i U_L \partial_\mu U_L^\dagger.\end{aligned}\tag{1}$$

Ω is the element of the nonlinearly represented $SU(2)_L$ gauge group

$$\Omega = \frac{1}{v_D}(\phi_0 + i\phi_a \tau_a), \quad \phi_0 = \sqrt{v_D^2 - \phi_a^2}.\tag{2}$$

v_D is a D -dimensional mass scale and τ_a are the Pauli matrices. $A_\mu = A_{a\mu} \frac{\tau_a}{2}$ is the $SU(2)_L$ gauge connection. It is also convenient to introduce the $SU(2)_L$ flat connection

$$F_\mu = F_{a\mu} \frac{\tau_a}{2} = i\Omega \partial_\mu \Omega^\dagger,\tag{3}$$

with the following $SU(2)_L$ transformation induced by the transformation of Ω

$$F'_\mu = U_L F_\mu U_L^\dagger + i U_L \partial_\mu U_L^\dagger.\tag{4}$$

¹the subscript L stands for the left action on the group element.

The amplitudes not involving the pseudo-Goldstone fields are named ancestor amplitudes since they are at the top of the hierarchy induced by the local functional equation. At every loop order there is only a finite number of divergent ancestor amplitudes (weak power-counting theorem [1],[3]). The requirement of physical unitarity is satisfied since the Slavnov-Taylor (ST) identity holds [1],[6]. The ghost equation and the Landau gauge equation are also preserved by the symmetric subtraction [1]. These symmetries, supplemented by global $SU(2)_R$ invariance and the weak power-counting, uniquely fix the tree-level vertex functional of the nonlinearly realized theory [1].

In [1] the Landau gauge was used for the sake of simplicity and conciseness. The aim of this note is to implement the 't Hooft α -gauge in a way compatible with all the symmetries required for the definition of the model (local functional equation, ST identity, ghost equation, B -equation for a general α -gauge [7]) and the weak power-counting.

The validity of the local functional equation requires that the gauge-fixing functional transforms in the adjoint representation of $SU(2)_L$. In the Landau gauge this was achieved by introducing an external vector source $V_{a\mu}$ and by making use of the gauge-fixing functional

$$\int d^D x B_a (D^\mu[V](A - V)_\mu)_a \quad (5)$$

where $(D_\mu[V])_{ac} = \partial_\mu \delta_{ac} + \epsilon_{abc} V_{b\mu}$ is the covariant derivative w.r.t. the vector source $V_{a\mu}$. B_a is the Nakanishi-Lautrup field [7]. It transforms in the adjoint representation of $SU(2)_L$. The local functional equation is preserved by the gauge-fixing (5).

It should be stressed that the local functional equation associated with the $SU(2)_L$ local invariance is not the standard background Ward identity [8]-[11]. The essential difference is that the local functional equation is bilinear in the vertex functional Γ , due to the presence of the nonlinear constraint ϕ_0 in eq.(2), which needs to be coupled to the scalar source K_0 in the tree-level vertex functional.

The 't Hooft α -gauge is defined by the condition of the cancellation (once the Nakanishi-Lautrup field is eliminated via its equation of motion) of the mixed $A_\mu \phi$ terms arising in the nonlinear theory from the mass invariant

$$\int d^D x \frac{M^2}{2} (A_{a\mu} - F_{a\mu})^2. \quad (6)$$

The 't Hooft (ϕ -dependent) gauge-fixing functional must transform in the adjoint representation of $SU(2)_L$ in order to preserve the local functional equation. For that purpose one needs to introduce an auxiliary matrix $\hat{\Omega}$

$$\hat{\Omega} = \frac{1}{v_D} (\hat{\phi}_0 + i \hat{\phi}_a \tau_a) \quad (7)$$

with the same $SU(2)_L$ transformation as Ω :

$$\hat{\Omega}' = U_L \hat{\Omega}. \quad (8)$$

The combination

$$\mathcal{F} = \mathcal{F}_a \frac{\tau_a}{2}, \quad \mathcal{F}_a = D^\mu[V](A - V)_{a\mu} + \frac{M^2}{2\alpha} Tr[i \hat{\Omega}^\dagger \tau_a \Omega + h.c.] \quad (9)$$

has the correct transformation properties. Therefore one can consider the following gauge-fixing functional

$$\int d^D x \left[-\frac{1}{2\alpha} B_a^2 + B_a \mathcal{F}_a \right], \quad (10)$$

where α is the gauge parameter. With the choice in eq.(10) the local functional equation is respected and by integrating the Nakanishi-Lautrup field B_a the mixed $A_\mu\phi$ -terms arising from eq.(6) are canceled. The propagators obtained by using the gauge-fixing functional in eq.(10) have a UV behaviour compatible with the weak power-counting.

It is important to realize that $\widehat{\Omega}$ is not an element of $SU(2)$. The reason is that the amplitudes involving $\widehat{\phi}_0$ and $\widehat{\phi}_a$ must be ancestors. Already at one loop level one cannot have a finite number of divergent amplitudes involving $\widehat{\phi}_a$ if $\widehat{\phi}_0$ is given by the $SU(2)$ constraint $\widehat{\phi}_0^2 + \widehat{\phi}_a^2 = v_D^2$. One must split in a linear way the constant component of $\widehat{\phi}_0$ by setting $\widehat{\phi}_0 \equiv v_D + \widehat{\sigma}$. Since $\widehat{\sigma}$ and $\widehat{\phi}_a$ are independent, by inspecting the Feynman rules one can then check that the weak power-counting is preserved. Moreover, since $\widehat{\phi}_0, \widehat{\phi}_a$ are independent variables, the BRST transformation can be extended to these sources by pairing them to external source ghosts θ_0, θ_a as follows

$$s\widehat{\phi}_0 = \theta_0, \quad s\theta_0 = 0, \quad s\widehat{\phi}_a = \theta_a, \quad s\theta_a = 0. \quad (11)$$

Then $(\widehat{\phi}_0, \theta_0), (\widehat{\phi}_a, \theta_a)$ form BRST doublets [7, 12] and therefore they do not contribute to the cohomology $H(s)$ of the BRST differential s . Hence they are not physical, as expected. The same technique can be used to prove that the vector source $V_{a\mu}$ does not modify the physical observables too [1].

The ghost-antighost part of the tree-level vertex functional is generated as usual by

$$\int d^D x s \left[\bar{c}_a \left(-\frac{1}{2\alpha} B_a + \mathcal{F}_a \right) \right] = \int d^D x \left(-\frac{1}{2\alpha} B_a^2 + B_a \mathcal{F}_a - \bar{c}_a s \mathcal{F}_a \right), \quad (12)$$

with the standard BRST transformation of the antighost \bar{c}_a , i.e. $s\bar{c}_a = B_a, sB_a = 0$.

2 Gauge dependence of the physical amplitudes

The issue arises of whether physical amplitudes depend on the gauge choice. This problem can be treated according to the standard extension of the BRST symmetry to the gauge parameter [7, 13]

$$s\alpha = \zeta, \quad s\zeta = 0. \quad (13)$$

The extended ST identity is in fact sufficient to prove the independence of the physical quantities from α also in the nonlinear case. We sketch here the main lines of the proof. Dropping inessential terms involving the background ghosts θ_0, θ_a the ST identity for the vertex functional Γ is²

$$\mathcal{S}(\Gamma) = \int d^D x \left[\Gamma_{A_{a\mu}^*} \Gamma_{A_a^\mu} + \Gamma_{\phi_a^*} \Gamma_{\phi_a} + \Gamma_{c_a^*} \Gamma_{c_a} + B_a \Gamma_{\bar{c}_a} \right] + \zeta \frac{\partial \Gamma}{\partial \alpha} = 0. \quad (14)$$

We define as usual the connected generating functional W by the Legendre transformation of Γ w.r.t. the quantized fields (collectively denoted by $\underline{\Phi}$)

$$W = \Gamma + \int d^D x \underline{K} \underline{\Phi}, \quad (15)$$

where \underline{K} is a short-hand notation for the sources of the quantized fields. Eq.(14) yields ($K(\varphi)$ stands for the source coupled to the field φ)

$$\mathcal{S}(W) = \int d^D x \left[-K(A_{a\mu}) W_{A_{a\mu}^*} - K(\phi_a) W_{\phi_a^*} - K(c_a) W_{c_a^*} - W_{K(B_a)} K(\bar{c}_a) \right] + \zeta \frac{\partial W}{\partial \alpha} = 0. \quad (16)$$

² Γ_X denotes the functional derivative of Γ w.r.t. X .

By differentiating eq.(16) w.r.t. ζ and a set of sources β_1, \dots, β_n coupled to physical BRST-invariant local operators $\mathcal{O}_1, \dots, \mathcal{O}_n$ one finds by going on-shell (all external sources set to zero)

$$\left. \frac{\partial}{\partial \alpha} W_{\beta_1 \dots \beta_n} \right|_{on-shell} = 0, \quad (17)$$

i.e. the physical Green function $W_{\beta_1 \dots \beta_n}$ is on-shell gauge-independent.

The Nielsen identities [14, 15] can also be obtained from the extended ST identity (14). We discuss here in detail the Nielsen identity for the two point 1-PI function of the gauge bosons.

By differentiating eq.(14) w.r.t. $A_{a_1\mu_1}, A_{a_2\mu_2}$ and ζ and by setting all the fields and external sources to zero one gets

$$\begin{aligned} & \Gamma_{\zeta A_{a\mu}^* A_{a_1\mu_1}} \Gamma_{A_{a_2\mu_2} A_{a\mu}} + \Gamma_{\zeta A_{a\mu}^* A_{a_2\mu_2}} \Gamma_{A_{a_1\mu_1} A_{a\mu}} + \\ & \Gamma_{\zeta \phi_a^* A_{a_1\mu_1}} \Gamma_{A_{a_2\mu_2} \phi_a} + \Gamma_{\zeta \phi_a^* A_{a_2\mu_2}} \Gamma_{A_{a_1\mu_1} \phi_a} + \partial_\alpha \Gamma_{A_{a_1\mu_1} A_{a_2\mu_2}} = 0. \end{aligned} \quad (18)$$

We decompose Γ_{AA} and $\Gamma_{\zeta A^* A}$ into their transverse and longitudinal components as follows

$$\begin{aligned} \Gamma_{A_{a\mu}(-p) A_{b\nu}(p)} &= \delta_{ab} \left(\Sigma_T^{AA}(p^2) T_{\mu\nu} + \Sigma_L^{AA}(p^2) L_{\mu\nu} \right), \\ \Gamma_{\zeta A_{a\mu}^*(-p) A_{b\nu}(p)} &= \delta_{ab} \left(\Sigma_T^{\zeta A^* A}(p^2) T_{\mu\nu} + \Sigma_L^{\zeta A^* A}(p^2) L_{\mu\nu} \right), \\ T_{\mu\nu} &= g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \quad L_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}. \end{aligned} \quad (19)$$

By taking the transverse part of eq.(18) one finds (notice that the terms proportional to $\Gamma_{A_\mu \phi}$ only contribute to the longitudinal part and thus drop out)

$$\partial_\alpha \Sigma_T^{AA}(p^2) = -2 \Sigma_T^{\zeta A^* A}(p^2) \Sigma_T^{AA}(p^2). \quad (20)$$

The self-mass \overline{M}^2 is defined as the zero of $\Sigma_T^{AA}(p^2)$:

$$\Sigma_T^{AA}(\overline{M}^2) = 0 \quad (21)$$

if no tadpoles are present as in the case under consideration. In passing it is worth noticing that the presence of tadpoles requires that the self-mass is defined as the pole of the transverse part of the connected two-point function (as it happens in the linearly realized theory).

By eq.(20) one finds

$$\partial_\alpha \Sigma_T^{AA}(\overline{M}^2) = 0. \quad (22)$$

Moreover invertibility of $\frac{\partial \Sigma_T^{AA}}{\partial p^2}$ is guaranteed in the loop expansion since

$$\frac{\partial \Sigma_T^{AA}}{\partial p^2} = -1 + O(\hbar).$$

The above equation together with eq.(22) implies

$$\frac{\partial \overline{M}^2}{\partial \alpha} = 0, \quad (23)$$

i.e. the self-mass is gauge-independent. This behaviour is a typical property of the nonlinear theory. In the linear case the zero of the two-point 1-PI function is in general gauge-dependent. Gauge independence can only be recovered by taking into account the Higgs tadpole contributions. For a detailed comparison of the two-point 1-PI function in the linear and the nonlinear case see Ref. [16].

3 Conclusions

The formulation of the nonlinearly realized $SU(2)$ massive gauge theory in the 't Hooft gauge has been achieved in a way consistent with all the symmetries of the model and the weak power-counting. This requires the introduction of auxiliary external sources $\hat{\sigma}, \hat{\phi}_a$. We have shown that this procedure does not alter the physical content of the model. Gauge independence of physical observables has been established by using cohomological methods. The self-mass, which can be computed in the nonlinearly realized theory as the zero of the transverse part of the 1-PI two-point function, has been proven to be gauge-independent.

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